# The hypercube and the core cover of N-person cooperative games

Tijs, S.H.; Lipperts, F.A.S.

*Published in:*
Cahiers du Centre d'Études de Recherche Opérationnelle

*Publication date:*
1982

# THE HYPERCUBE AND THE CORE COVER OF N-PERSON COOPERATIVE GAMES

S.H. TIJS and F.A.S. LIPPERTS

Department of Mathematics
Catholic University, Nijmegen
The Netherlands

ABSTRACT

For n-person games in characteristic function form two subsets of $\mathbb{R}^n$ are introduced, the hypercube and the core cover, and many interesting properties of these sets are derived. The core cover contains the core of a game and for various subclasses of games it coincides with the core. A characterization of the games with a non-empty hypercube is given in terms of balanced sets of a special type.

KEYWORDS

Games in characteristic function form, core cover, hypercube, semi-balanced games.

## 1. INTRODUCTION AND SUMMARY

This paper deals with n-person games in characteristic function form. In section 2 we summarize some definitions and notations. In section 3 we start with giving upper and lower bounds for the coordinates of the imputations which are in the core of a game, which bounds are sharp for many games and can easily be calculated. With the aid of these bounds we define two subsets of $\mathbb{R}^n$ : the hypercube and the core cover of a game. These subsets behave well with respect to strategic equivalence, symmetry, the presence of dummy players, perturbations of the characteristic function, etc.

In general the core of a game is a subset of the core cover. The core coincides with the core cover for some classes of games, e.g. 2- and 3-person games, constant sum games and simple monotone games. The obtained results may be useful in determining cores. A necessary condition for the non-emptyness of the core is the non-emptyness of the hypercube. Balanced families of subsets of the players set of a special type play a role in the characterization of the class of games with a non-empty hypercube. To be more concrete, in section 4 we introduce the concept of semi-balancedness and we prove that the hypercube of a game is non-empty iff the game is semi-balanced. It appears that the families of characteristic functions of games with a non-empty hypercube and also those with a non-empty core cover form polyhedral cones of full dimension in the space of all characteristic functions of n-person games.

## 2. PRELIMINARIES

In the following n is a natural number greater than 1, $N = \{1, 2, \ldots, n\}$ and $2^N$ is the family of all subsets of N. An *n-person game (in characteristic function form)* is a pair $\langle N,v \rangle$, where v is a realvalued function on $2^N$ with $v(\emptyset) = 0$. The elements of N are called *players* and the elements of $2^N$ *coalitions*. The function v is called the *characteristic function* and for $S \in 2^N$, the real number $v(S)$ is called the *worth* or *value* of the coalition S. We shall call a game $\langle N,v \rangle$

- an *additive game* if $v(S \cup T) = v(S) + v(T)$ for all disjoint pairs $S,T \in 2^N$
- a *superadditive game* (s.a. game) if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint pairs $S,T \in 2^N$
- a *convex game* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S,T \in 2^N$ (1)
- a *constant sum game* if $v(S) + v(N-S) = v(N)$ for all $S \in 2^N$
- a *simple game* if $v(S) \in \{0,1\}$ for all $S \in 2^N$
- a *monotone game* if $S \subset T$ implies $v(S) \leq v(T)$ for $S,T \in 2^N$
- a *symmetric game* if $v(S) = v(T)$ for all $S,T \in 2^N$ with $|S| = |T|$.

(Here $|S|$ is the number of elements in S).
For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $S \in 2^N - \{\emptyset\}$ the sum $\sum_{i\in S} x_i$ will also be denoted by $x(S)$ and $x(\emptyset) = \sum_{i\in\emptyset} x_i = 0$.
For an n-person game $\langle N,v \rangle$ the set $I(v) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\})$ for each $i \in N$, $x(N) = v(N)\}$ is called the *set of imputations* and the subset

$$C(v) = \{x \in I(v) : x(S) \geq v(S) \text{ for each } S \in 2^N\}$$

the *core* of the game.
If for two n-person games $\langle N,v \rangle$ and $\langle N,w \rangle$ there exist $k \in (0,\infty)$ and $c = (c_1,c_2,\ldots,c_n) \in \mathbb{R}^n$ such that

$$w(S) = kv(S) + c(S) \text{ for each } S \in 2^N,$$

then we say that $\langle N,w \rangle$ is *strategic equivalent* with $\langle N,v \rangle$ via $(k;c_1,c_2,\ldots,c_n)$.
A player i in a game $\langle N,v \rangle$ is called a *dummy player* if

$$v(S \cup \{i\}) = v(S) + v(\{i\}) \text{ for each } S \in 2^N \text{ with } i \notin S.$$

A family $\mathcal{B}$ of subsets of $2^N - \{\emptyset\}$ is called a *balanced family* if there exists a (weight) function $\omega : 2^N \to [0,\infty)$, such that $\omega(S) > 0$ iff $S \in \mathcal{B}$ and such that $1_N = \sum_{S\in 2^N} \omega(S)1_S$; here $1_S : N \to \{0,1\}$ is the function with $1_S(i) = 1$ iff $i \in S$.
A game $\langle N,v \rangle$ is called a *balanced game* if for each balanced family $\mathcal{B}$ of subsets of $2^N - \{\emptyset\}$ with weight function $\omega$ we have

$$\sum_{S\in\mathcal{B}} \omega(S)\, v(S) \leq v(N).$$

From now on we will often write $v(i)$, $v(i,j)$, $S-i$, $x(S-i),\ldots,$ instead of $v(\{i\})$, $v(\{i,j\})$, $S - \{i\}$, $x(S - \{i\})$, ... .

## 3. THE HYPERCUBE AND THE CORE COVER OF A GAME

Let $\langle N,v \rangle$ be an n-person game. We define $b_i^v = v(N) - v(N-i)$ for each $i \in N$, $b^v = (b_1^v, b_2^v, \ldots, b_n^v)$; $R^v(S,i) = v(S)-b^v(S-i)$ for each $S \in 2^N$ and $i \in S$ (where $b^v(S-i) = \sum_{j\in S-i} b_j^v$ and $b^v(\emptyset) = 0$); $a_i^v = \max\left\{R^v(S,i) : S \in 2^N, i \in S\right\}$ and $a^v = (a_1^v,a_2^v,\ldots,a_n^v)$.

We call the number $b_i^v$ the *marginal value of player* i in the game. It may be interpreted as the bargaining power of player i with respect to the coalition N, because it is equal to the increase of worth caused by i when that player enters the coalition N-i.
We call the number $R^v(S,i)$ the *remainder in the coalition* S *for player* i; it is the payoff which can be made to player i in the coalition S when the other players in this coalition obtain their marginal value.
The number $a_i^v$ is the *maximal remainder for player* i. Now we introduce two interesting subsets of $\mathbb{R}^n$.

DEFINITIONS

We call the set $\{x \in \mathbb{R}^n : a_i^v \leq x_i \leq b_i^v$ for each $i \in N\}$ the *hypercube* of the game $\langle N,v \rangle$ and we denote it by $H(v)$. The set $\{x \in H(v) : \sum_{i=1}^{n} x_i = v(N)\}$ will be called the *core cover* of the game and it will be denoted by $CC(v)$.

Without proof we give some simple properties

1. $a_i^v \geq R^v(\{i\}, i) = v(i)$.
2. $CC(v) \subset I(v)$.
3. $H(v)$ and $CC(v)$ are compact and convex sets (even polytopes).
4. If $\langle N,v \rangle$ is a symmetric game, then all coordinates of $a^v$ and of $b^v$ are equal; $CC(v) \neq \emptyset$ iff $a_i^v \leq n^{-1}\, v(N) \leq b_i^v$.
5. If $\langle N,v \rangle$ is an additive game, then $H(v) = C(v) = CC(v) = \{(v(1), v(2), \ldots, v(n))\}$.

The name core cover is explained by the following

THEOREM 1. For each game $\langle N,v \rangle$ we have $C(v) \subset CC(v) \subset H(v)$.
PROOF. Let $x \in C(v)$. Then for each $i \in N$ we have

(a) $x_i = x(N) - x(N-i) = v(N) - x(N-i) \leq v(N) - v(N-i) = b_i^v$

(b) for each S with $i \in S$ we have
$$x_i = x(S) - x(S-i) \geq v(S) - x(S-i) \geq v(S) - b^v(S-i) = R^v(S,i),$$
where the last inequality follows from (a). But then $x_i \geq a_i^v$.
Hence $a^v \leq x \leq b^v$, $x \in H(v)$.
Since $C(v) \subset I(v)$, we have $x(N) = v(N)$. So $x \in CC(v)$ and we have proved that $C(v) \subset CC(v)$.
The other inclusion in the theorem is trivial. □

REMARKS

1. Theorem 1 may be useful to decide whether the core of a game is empty. If e.g. $a_i^v > b_i^v$ for some $i \in N$, then $H(v) = \emptyset$ and also $C(v)$. The non-emptyness of $H(v)$ is a necessary condition for the non-emptyness of the core, but not a sufficient condition as example 2 below shows. If $a(N) > v(N)$, then also $C(v) = \emptyset$; and that is also true if $b^v(S) < v(S)$ for some $S \in 2^N$.

2. In section 4 a necessary and sufficient condition for the non-emptyness of $H(v)$ is given.

3. The numbers $a_i^v$ and $b_i^v$, respectively are, in view of theorem 1, lower and upper bounds

for the i-th coordinate of core elements. The question arises whether they are sharp. For an answer look at the examples 2 and 4 below and at the theorems 2-6, in which classes of games are considered, where these bounds are sharp.

4. In example 3 below we give an example of a game for which $H(v) \neq \emptyset$ and $CC(v) = \emptyset$.

EXAMPLES

1. Let $\langle N,v \rangle$ be the symmetric convex 4-person game with

$$v(S) = 0, \tfrac{1}{4}, \tfrac{1}{2}, 1 \text{ if } |S| = 1, 2, 3, 4, \text{ respectively.}$$

Then $H(v) = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.
This example shows that $C(v)$ may be a proper subset of $CC(v)$; note that $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ is an element of $CC(v)$ and not of $C(v)$.

2. Let $\langle N,v \rangle$ be the monotone 4-person game with

$$v(S) = 0, \tfrac{5}{8}, \tfrac{5}{8}, 1 \text{ if } |S| = 1, 2, 3, 4, \text{ respectively.}$$

Then $b^v = (\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8},)$ and $a^v = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. So $H(v) \neq \emptyset$.
In this example $C(v)$ is empty and $CC(v) = \{a^v\}$.

3. Now we give an example for which $H(v) \neq \emptyset$ and $CC(v) = \emptyset$. Let $\langle N,v \rangle$ be the symmetric 4-person game with

$$v(S) = 0, \tfrac{2}{3}, \tfrac{2}{3}, 1 \text{ if } |S| = 1, 2, 3, 4, \text{ respectively.}$$

Then $H(v) = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\} \not\subset I(v)$, hence $CC(v) = \emptyset$.

4. For the 4-person game $\langle N,v \rangle$ with

$$v(S) = 0, \tfrac{1}{2}, \tfrac{2}{3}, 1 \text{ if } |S| = 1, 2, 3, 4, \text{ respectively}$$

we have $a^v = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, $b^v = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $C(v) = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$.

DEFINITIONS

We shall say that the game $\langle N,v \rangle$ is *rigid*, if $H(v)$ is empty or if
$\forall_{i \in N} \exists_{x,y \in CC(v)} [x_i = a_i^v, y_i = b_i^v]$.
We shall say that the game $\langle N,v \rangle$ is *ideally covered*, if $C(v) = CC(v)$.

Note that for an ideally covered rigid game the numbers $a_i^v$ and $b_i^v$ are sharp lower and upper bounds, respectively, for the i-th coordinate of core elements. For such games the hypercube $H(v)$ is the smallest hypercube containing $C(v)$, with facets parallel to the coordinate hyperplanes. The games in examples 1 and 4 are rigid and not ideally covered; the game in example 2 is not rigid and not ideally covered; the game in example 3 is ideally covered and not rigid.

In the following four theorems, classes of games which are rigid and ideally covered are considered.

THEOREM 2. Each 2-person game is rigid and ideally covered.
PROOF. Let $\langle N,v \rangle$ be a two-person game. We consider three cases.

1. If $v(N) < v(1) + v(2)$, then $b^v < a^v$. Hence $H(v) = C(v) = CC(v) = \emptyset$.
2. If $v(N) = v(1) + v(2)$, then the game is an additive game and then $C(v) = CC(v) = H(v) = \{(v(1), v(2))\}$.
3. If $v(N) > v(1) + v(2)$, then $H(v) = [v(1), v(N) - v(2)] \times [v(2), v(N) - v(1)]$ and $CC(v)$, $C(v)$ and $I(v)$ coincide with the line segment in $\mathbb{R}^2$ with endpoints $(v(1), v(N) - v(1))$ and $(v(N) - v(2), v(2))$.

Obviously, in all three cases the game $< N,v >$ is rigid and ideally covered. □

THEOREM 3. Each 3-person game is ideally covered and each super-additive 3-person game is also rigid.

PROOF. Let $< N,v >$ be a 3-person game.

1. Let $x \in CC(v)$. Then $x \in I(v)$ and $x_1 + x_2 = v(N) - x_3 \geq v(N) - b_3^v = v(N) - (v(N) - v(\{1,2\})) = v(1,2)$, $x_1 + x_3 \geq v(1,3)$ and $x_2 + x_3 \geq v(2,3)$. Hence $x \in C(v)$. In view of theorem 1, we have proved that $< N,v >$ is ideally covered.

2. Now suppose that $< N,v >$ is superadditive. Let $S_1 = \{2,3\}$, $S_2 = \{1,3\}$ and $S_3 = \{1,2\}$. We distinguish two cases.

2.1. First suppose that $\sum_{i=1}^{3} v(S_i) > 2v(N)$. Then

$$a_i^v \geq v(N) - b^v(S_i) = v(N) - \sum_{k \neq i} (v(N) - v(S_k)) > v(N) - v(S_i) = b_i^v.$$

Hence $H(v) = \emptyset$ and thus $< N,v >$ is rigid.

2.2. Now suppose that $\sum_{i=1}^{3} v(S_i) \leq 2v(N)$. Then

$$a_i^v = \max \{v(i), - v(N) + \sum_{k \neq i} v(S_k)\} \ (\leq v(N) - v(S_i) = b_i^v).$$

If we can show that the points

$(a_1^v, b_2^v, v(N) - a_1^v - b_2^v)$, $(v(N) - a_2^v - b_3^v, a_2^v, b_3^v)$ and $(b_1^v, v(N) - a_3^v - b_1^v, a_3^v)$ are elements of $C(v) \subset CC(v)$, then we may conclude that $< N,v >$ is rigid. We consider only the first point $x = (a_1^v, b_2^v, v(N) - a_1^v - b_2^v)$.

Note that $x_1 = a_1^v \geq v(1)$ and that by superadditivity

$x_2 = b_2^v = v(N) - v(1,3) \geq v(2)$. Moreover,

$x_3 = v(N) - a_1^v - b_2^v = v(1,3) - a_1^v$, where $v(1,3) - a_1^v \geq v(3)$ if $a_1^v = v(1)$; and if $a_1^v \neq v(1)$, then

$$v(1,3) - a_1^v = v(1,3) + v(N) - \sum_{k \neq 1} v(S_k) = v(N) - v(1,2) \geq v(3).$$

So $x_i \geq v(\{i\})$ for each $i \in N$. Because also $x(N) = v(N)$, we may conclude that $x \in I(v)$. To prove that $x \in C(v)$, we distinguish two subcases.

2.2.1. First, let $a_1^v = v(1) \geq -v(N) + v(S_2) + v(S_3)$. Then we have

$x_1 + x_2 = a_1^v + b_2^v = v(1) + v(N) - v(S_2) \geq v(S_3) = v(1,2)$,

$x_1 + x_3 = a_1^v + v(N) - a_1^v - b_2^v = v(N) - b_2^v = v(S_2) = v(1,3)$ and

$x_2 + x_3 = v(N) - a_1^v = v(N) - v(1) \geq v(2,3)$ by superadditivity.

Hence $x \in C(v)$ in this subcase.

2.2.2. Secondly, let $a_1^v = -v(N) + v(S_2) + v(S_3) \geq v(1)$. Then we have

$x_1 + x_2 = a_1^v + b_2^v = -v(N) + v(S_2) + v(S_3) + v(N) - v(S_2) = v(1,2)$,

$x_1 + x_3 = v(1,3)$ as in (2.2.1.) and

$x_2 + x_3 = v(N) - a_1^v = v(N) + v(N) - v(S_2) - v(S_3) \geq v(S_1) = v(2,3)$.

So $x \in C(v)$ also in this subcase. □

It follows from the following theorem that monotone simple games are ideally covered and rigid. Furthermore, in part 2 of this theorem a characterization of the core and core cover elements for such games is given.

THEOREM 4. Let $\langle N,v\rangle$ be a monotone simple game and let $J = \{i \in N : b_i^v = 1\}$. Then we have

1. $J = \cap \{S \in 2^N : v(S) = 1\}$.
2. $CC(v) = C(v) = \{x \in I(v) : x(J) = v(N)\}$.

3.0. If $v(N) = 0$, then $H(v) = \{0\}$.

3.1. If $J = \emptyset$ and $v(N) = 1$, then $H(v) = \emptyset$.

3.2. If $J = \{j\}$ for some $j \in N$, then $H(v) = \{e^j\}$ (where $e^j$ is the j-th standard basis vector in $\mathbb{R}^n$).

3.3. If $|J| \geq 2$, then $H(v) = \{x \in \mathbb{R}^n : x_i = 0$ for $i \notin J$ and $0 \leq x_i \leq 1$ for $i \in J\}$.

PROOF.

1. Let $i \in J$. Then $v(N) = 1$ and $v(N - \{i\}) = 0$. By monotonicity, for each $T \subset N - \{i\}$ we have $v(T) = 0$. Hence $i \in \cap \{S \in 2^N : v(S) = 1\}$.
   Conversely, let $i \in \cap \{S \in 2^N : v(S) = 1\}$. Then $v(N) = 1$ and $v(N-i) = 0$ since $i \notin N - \{i\}$. So $b_i^v = 1$, $i \in J$.

2. We will prove the equalities in part 2 by verifying the following string of inclusions :
   $\{x \in I(v) : x(J) = x(N)\} \subset C(v) \subset CC(v) \subset \{x \in I(v) : x(J) = x(N)\}$.
   Let $x \in I(v)$, $x(J) = x(N)$. Then for each $S \in 2^N$ with $v(S) = 0$ we have
   $x(S) \geq \sum_{i \in S} v(i) \geq 0 = v(S)$. For an $S \in 2^N$ with $v(S) = 1$ we have $J \subset S$ in view of part 1, and then
   $x(S) \geq x(J) = x(N) = 1 = v(S)$.
   So $x \in C(v)$, and the first inclusion in the string is proved. The second inclusion follows from theorem 1. Now let $x \in CC(v)$. Then $x \in I(v)$. For $i \notin J$ we have $b_i^v = 0$ and so
   $0 \leq v(i) \leq x_i \leq b_i^v = 0$.
   Hence $x_i = 0$. Then $x(J) = x(N) = v(N)$, and also the third inclusion in the string is proved.

3. Let $S \in 2^N$ and $i \in S$. Then
   $R^v(S,i) = v(S) - b^v(S-i) = v(S) - |(S - \{i\}) \cap J|$ (2)
   From (2) and part 1 of the theorem we may conclude that
   $R^v(S,i) \leq 0$ if $v(S) = 0$, (3)
   $R^v(S,i) = 1 - |J|$ if $v(S) = 1$ and $i \notin J$, (4)
   $R^v(S,i) = 2 - |J|$ if $v(S) = 1$ and $i \in J$. (5)

3.0. follows immediately from (3).

3.1. If $J = \emptyset$ and $v(N) = 1$, then $R^v(N,i) = 1$. So $a_i^v = 1 > 0 = b_i^v$ and thus $H(v) = \emptyset$.

3.2. Let $J = \{j\}$. Then $a_i^v = b_i^v = 0$ for each $i \in N - \{j\}$ in view of (4) and $a_j^v = b_j^v = 1$ in view of (5). So $H(v) = \{e^j\}$.

3.3. Let $|J| \geq 2$. By (4) and (5), we have $a_i^v = 0$ for each $i \in N$. So
$H(v) = \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i \notin J \text{ and } 0 \leq x_i \leq 1 \text{ for } i \in J\}$. □

Now we prove a lemma, which appears to be useful in the following.

LEMMA 1. Let $< N,v >$ be a game such that $H(v) \neq \emptyset$ and $b^v(N) = v(N)$. Then $H(v) = C(v) = CC(v) = \{b^v\}$.

PROOF. Since $H(v) \neq \emptyset$, we have $b_i^v \geq a_i^v \geq v(i)$ for each $i \in N$. Because $b^v(N) = v(N)$, we may conclude that $b^v \in I(v)$. Now we want to prove that $a^v = b^v$. Take $i,j \in N$, $i \neq j$. Then $a_i^v \geq v(N-j) - \sum_{k \neq i,j} b_k^v = v(N-j) - (v(N) - b_i^v - b_j^v) = b_i^v$. Hence $a_i^v = b_i^v$ for each $i \in N$. So $H(v) = \{b^v\}$ and $CC(v) = H(v) \cap I(v) = \{b^v\}$.

In view of theorem 1, the only thing that remains is to prove that $b^v \in C(v)$. For each $S \in 2^N - \{\emptyset\}$ and each $i \in S$ we have

$$b^v(S) = b_i^v + b^v(S-i) = a_i^v + b^v(S-i) \geq (v(S) - b^v(S-i)) + b^v(S-i) = v(S).$$

Hence $b^v \in C(v)$. □

REMARKS

1. Note that property 5, concerning additive games at the beginning of this section can easily be derived from the foregoing lemma.

2. In the paper of R. Spinetto [7] the n-person games $v_k$ ($n \geq 3$, $k \in N$) defined by $v_k(S) = 1$ if $k \in S$ and $|S| > 1$ and $v_k(S) = 0$ otherwise, play an important role. Now $b^{v_k} = \{e^k\}$, $b^{v_k}(N) = 1 = v(N)$. So it follows immediately from lemma 1 that $CC(v_k) = C(v_k) = H(v_k) = \{e^k\}$.

3. Look at the "glove game" $< N,v >$ (cf. [4], p. 13), where $\{R,L\}$ is a partition of N in two non-empty sets R and L with $|R| > |L|$, and where $v(S) = \min\{|S \cap L|, |S \cap R|\}$. Then $b_i^v = 1$ if $i \in L$ and $b_i^v = 0$ if $i \in R$. Hence, $b^v(N) = |L| = v(N)$. So $CC(v) = C(v) = \{b^v\}$ in view of lemma 1.

4. The lemma will also be used in the following theorem, concerning constant sum games. Those games appear to be rigid and ideally covered. Furthermore, the core cover and the core of superadditive constant sum games are empty sets (cf. [8], p. 138).

THEOREM 5. Let $< N,v >$ be a constant sum game. Then
$b^v = (v(1), v(2), \ldots, v(n))$. Furthermore,

1. $H(v) \neq \emptyset$ iff $v(S) \leq \sum_{j \in S} v(j)$ for each $S \in 2^N$.
2. If $H(v) \neq \emptyset$ and $v(N) = \sum_{i \in N} v(i)$, then $H(v) = CC(v) = C(v) = I(v) = \{b^v\}$.
3. If $H(v) \neq \emptyset$ and $v(N) < \sum_{i \in N} v(i)$, then $CC(v) = C(v) = I(v) = \emptyset$.

PROOF. $b_i^v = v(N) - v(N-i) = v(i)$ for each $i \in N$, since $< N,v >$ is a constant sum game

1. $H(v) \neq \emptyset \Leftrightarrow a_i^v \leq b_i^v$ for each $i \in N$
$\Leftrightarrow v(S) - b^v(S-i) \leq b_i^v$ for each $i \in N$ and each $S \ni i$
$\Leftrightarrow v(S) \leq \sum_{i \in S} v(i)$ for each $S \in 2^N$.

2. follows immediately from lemma 1.

3. If $v(N) < \sum_{i \in N} v(i)$, then $I(v) = \emptyset$ and then also $C(v) = CC(v) = \emptyset$. $\square$

That convex games are rigid can be seen from the following theorem.

THEOREM 6. Let $\langle N,v \rangle$ be a convex game. Then

1. $a_i^v = v(i)$ for each $i \in N$.
2. For all $k,l \in N$, $k \neq l$, there is an extreme point $x$ of $C(v)$, such that $x_k = a_k^v$ and $x_l = b_l^v$.

PROOF.

1. For a convex game we have (cf. formula (6) on p. 13 in Shapley [6]) :

$v(S) - v(S-i) \leqslant v(N) - v(N-i) = b_i^v$ for each $S \in 2^N$ and $i \in S$.

Using $|R|$ of such inequalities, it is straightforward to show that

$$v(N) - v(N-R) \leqslant b^v(R) \text{ for each } R \in 2^N - \{\emptyset\} \tag{6}$$

Let $i \in N$. Take $S^* \in 2^N$ with $i \in S^*$ and $a_i^v = v(S^*) - b^v(S^* - i)$. Then

$$\begin{aligned} a_i^v = v(S^*) - b^v(S^* - i) &\leqslant v(S^*) + v(N - (S^* - i)) - v(N) \\ &\leqslant v(N) + v(i) - v(N) = v(i), \end{aligned}$$

where the first inequality follows from (6) with $S^* - \{i\}$ in the role of $R$, and where the second inequality follows from (1) with $S^*$ and $N - (S^* - i)$ in the roles of $S$ and $T$, respectively. Since also $a_i^v \geqslant v(i)$, we may conclude that $a_i^v = v(i)$ for each $i \in N$.

2. Now let $k$ and $l$ be two elements of $N$. We construct a map $\omega$ from $N$ onto $N$, such that $\omega(k) = 1$ and $\omega(l) = n$. Shapley proved in section 3.3. of [6] that the point $x$ in $\mathbb{R}^n$ with i-th coordinate

$$x_i = v(\{j \in N : \omega(j) \leqslant \omega(i)\}) - v(\{j \in N : \omega(j) \leqslant \omega(i) - 1\}) \qquad (i \in N),$$

is an extreme point of $C(v)$. Now $x_k = v(k) - v(\emptyset) = a_k^v$ in view of part 1, and $x_l = v(N) - v(N-l) = b_l^v$. $\square$

Now let $\langle N,v \rangle$ and $\langle N,w \rangle$ be n-person games and let $k \in (0,\infty)$ and $(c_1,c_2,\ldots,c_n) \in \mathbb{R}^n$. Then the following facts are easy to show.

1. $b^{v+w} = b^v + b^w$, $b^{kv} = kb^v$.
2. $R^{v+w}(S,i) = R^v(S,i) + R^w(S,i)$, $R^{kv}(S,i) = kR^v(S,i)$ if $S \in 2^N$ and $i \in S$.
3. $a^{v+w} \leqslant a^v + a^w$, $a^{kv} = ka^v$.
4. $H(v+w) \supset H(v) + H(w)$, $H(kv) = kH(v)$.
   [$H(v) + H(w) = \{x+y : x \in H(v), y \in H(w)\}$, $kH(v) = \{kx : x \in H(v)\}$.]
5. $CC(v+w) \supset CC(v) + CC(w)$, $CC(kv) = kCC(v)$.
6. If $w$ is additive, then $a^{v+w} = a^v + (w(1), \ldots, w(n))$, $H(v+w) = H(v) + H(w)$, $CC(v+w) = CC(v) + CC(w)$.

From these facts the next two theorems follow immediately
The first of these says that the two solution concepts core cover and hypercube behave well with respect to strategic equivalence.

THEOREM 7. Let $\langle N,w \rangle$ be strategic equivalent with $\langle N,v \rangle$ via $(k;c_1,\ldots,c_n)$. Then $H(w) = kH(v) + c$ $(= \{kx+c : x \in H(v)\})$ and $CC(x) = kCC(v) + c$.

Let us denote the $(2^n-1)$-dimensional space of characteristic functions of n-person games by $G_n$ and let $H_n$ be the set of elements of $G_n$ with non-empty hypercube, and $K_n = \{v \in G_n : CC(v) \neq \emptyset\}$. Then in view of properties 4 and 5 above we have :

THEOREM 8. $H_n$ and $K_n$ are convex cones.

The dimension of these cones is equal to $2^n-1$, because they both contain (cf. theorem 4) the set $\{u_T : T \in 2^N - \{\emptyset\}\}$ of $2^n-1$ linearly independent (monotone simple) characteristic functions, where

$$u_T(S) = 1 \text{ if } S \supset T \text{ and otherwise } u_T(S) = 0.$$

In the next section we prove that these cones are polyhedral.

Now we look at the influence of perturbations of the characteristic function on the hypercube and the core cover. Therefore we put on $G_n$ the metric d, defined by

$$d(v,w) = \max \{|v(S) - w(S)| : S \in 2^N - \{\emptyset\}\} \text{ for } v,w \in G_n.$$

We note that

$$\|b^v - b^w\|_\infty \leqslant 2d(v,w) \quad [\|x\|_\infty = \max_{i \in N} x_i \text{ for } x \in \mathbb{R}^n.] \tag{7}$$

$$R^v(S,i) \leqslant R^w(S,i) + (2|S| - 1)\, d(v,w) \text{ if } S \in 2^N \text{ and } i \in S \tag{8}$$

$$\|a^v - a^w\|_\infty \leqslant (2n - 1)\, d(v,w) \tag{9}$$

Let now $H_n^* = \{v \in G_n : a_i^v < b_i^v \text{ for all } i \in N\}$,

$K_n^* = \{v \in H_n^* : a^v(N) < v(N) < b^v(N)\}$.

Note that $H_n^* \subset H_n$ and $K_n^* \subset K_n$.
From (7) and (9) we obtain easily

THEOREM 9. $H_n^*$ and $K_n^*$ are open subsets of $G_n$. The multifunction $v \to H(v)$ from $H_n^*$ into $\mathbb{R}^n$ is a continuous multifunction (cf. C. Berge [1]).
The multifunction $v \to CC(v)$ from $K_n^*$ into $\mathbb{R}^n$ is continuous.

## 4. *SEMI-BALANCED GAMES*

A combinatorial characterization of games with a non-empty core was given by O.N. Bondareva [2] and L.S. Shapley [5] and a geometric characterization by R.D. Spinetto [7] (cf. G. Bruyneel [3], chapter 5). The core of a game appeared to be non-empty iff the game is balanced. Furthermore, the set of characteristic functions of games with a non-empty core appeared to form a polyhedral cone.
Now we want to characterize the cone $H_n$ of games with a non-empty hypercube and to prove that $H_n$ is a polyhedral cone. Therefore we need the following

DEFINITION. We shall call a game $\langle N,v \rangle$ a *semi-balanced game* if the following system of inequalities holds :

$$v(S) + \sum_{i \in S} v(N-i) \leqslant |S|\, v(N) \text{ for each } S \in 2^N - \{\emptyset\}. \tag{10}$$

Note that for each $S \in 2^N - \{\emptyset\}$ the family of $|S| + 1$ coalitions

$\{S, (N - \{i\})_{i \in S}\}$ is a balanced family since

$$|S|^{-1} 1_S + \sum_{i \in S} |S|^{-1} 1_{N-\{i\}} = 1_N.$$

Hence, the following theorem is obvious.

THEOREM 10. Each balanced game is semi-balanced.

Now 2- and 3-person semi-balanced games are also balanced, but this is not true for all n-person games as example 2 in section 3 shows. The semi-balancedness property appears to be a necessary and sufficient condition for the non-emptyness of the hypercube.

THEOREM 11. $v \in H_n$ iff $< N,v >$ is a semi-balanced game.
$H_n$ is a polyhedral cone.

PROOF. $v \in H_n \Leftrightarrow a_i^v \leq b_i^v$ for each $i \in N$

$\Leftrightarrow v(S) - b^v(S-i) \leq b_i^v$ for each $i \in N$ and each $S \ni i$

$\Leftrightarrow v(S) \leq b^v(S)$ for each $S \in 2^N - \{\emptyset\} \Leftrightarrow (10)$.

Hence $H_n$ is characterized by the inequalities in (10), which are finite in number and thus $H_n$ is polyhedral. □

The proof of the following theorem is left to the reader.

THEOREM 12. If in a game $< N,v >$ with $v \in H_n$ player i is a dummy player, then $a_i^v = b_i^v = v(i)$.

Now we want to prove that $K_n$ is also a polyhedral cone.
We note that

$$K_n = \{v \in H_n : a^v(N) \leq v(N) \leq b^v(N)\}. \tag{11}$$

Now

$$v(N) \leq b^v(N) \text{ iff } \sum_{i \in N} v(N-i) \leq (n-1)\, v(N) \tag{12}$$

and

$$a^v(N) \leq v(N) \text{ iff } \sum_{i \in N} v(S_i) - b^v(S_i - i) \leq v(N) \text{ or}$$

$$\sum_{i \in N} (v(S_i) + \sum_{j \in S_i - i} v(N-j)) \leq (1 - n + \sum_{i \in N} |S_i|)\, v(N) \tag{13}$$

for all n-tuples $(S_1, S_2, \ldots, S_n)$ with $S_i \in 2^N$ and $i \in S_i$ $(i \in N)$.
From (11), (12), (13) and theorem 11 we may conclude :

THEOREM 13. $K_n$ is a polyhedral cone.

## REFERENCES


[1] BERGE, C. (1959). *Espaces topologiques*. Dunod, Paris.

[2] BONDAREVA, O. (1962). The core of an n-person game. *Vestnik Leningrad University*, 17, 141-142. (English translation : Selected Russian papers on game theory, 1959-65, Economic Research Program, Princeton University, Princeton, 1968, 29-31).

[3] BRUYNEEL, G. (1978). *Gebalanceerde incidentiestructuren en de toepassingen ervan in de theorie der cooperatieve n-personenspelen.* Thesis, University of Gent, Belgium.

[4] ROSENMULLER, J. (1971). *Kooperative Spiele und Märkte.* Springer Verlag, Berlin.

[5] SHAPLEY, L.S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly, 14*, 453-460.

[6] SHAPLEY, L.S. (1971). Cores of convex games. *Intern. J. of Game Theory, 1*, 11-26.

[7] SPINETTO, R. (1974). The geometry of solution concepts for n-person cooperative games. *Management Science, 20*, 1292-1299.

[8] VOROBEV, N.N. (1977). *Game Theory, Lectures for Economists and Systems Scientists.* Springer Verlag, New York - Berlin - Heidelberg.